STAT1012 Statistics for Life Sciences

Quick Revision Notes

Spring, 2020

LEUNG Man Fung, Heman

(Reference: lecture and tutorial notes)

Contents

I) Descriptive Statistics

Data type: Qualitative (special: Categorical), Quantitative (Discrete, Continuous)

Population: the whole set of entities of interest

Sample: a subset of the population

Central tendency

Sample mean: $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

Sequential update property: $\bar{X}_n = \frac{1}{n}[(n-1)\bar{X}_{n-1} + X_n]$

Mode: the value which has the greatest number of occurrence (may not be unique)

Median: the "middle" value, or the average of the two values closest to "middle" after sorting

Percentile: the p-th percentile ($V_{\frac{p}{100}}$) is a value such that p% of the data are less than or equal to $V_{\frac{p}{100}}$. In particular, upper quantile = $V_{0.75}$, median = $V_{0.5}$, lower quantile = $V_{0.25}$

Denote the sorted data by $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ where $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$. This is equivalent to saying that $X_{(1)}$ is the smallest, $X_{(2)}$ is the second smallest etc.

Median: $V_{0.5} = X_{\left(\frac{n+1}{2}\right)}$ if n is odd or $\frac{1}{2}\left[X_{\left(\frac{n}{2}\right)} + X_{\left(\frac{n}{2}+1\right)}\right]$ if n is even

Percentile: $V_{\frac{p}{100}} = X_{(k)}$ where $k = \text{roundUp}\left(\frac{np}{100}\right)$ if $\frac{np}{100}$ is not an integer

Otherwise, $V_{\frac{p}{100}} = \frac{1}{2}\left[X_{\left(\frac{np}{100}\right)} + X_{\left(\frac{np}{100}+1\right)}\right]$

Dispersion

Symmetric: the left hand side of the distribution mirrors the right hand side

Unimodal: the mode is unique

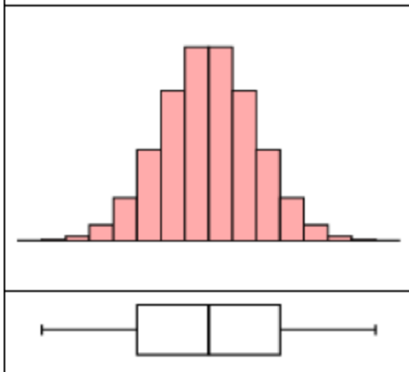
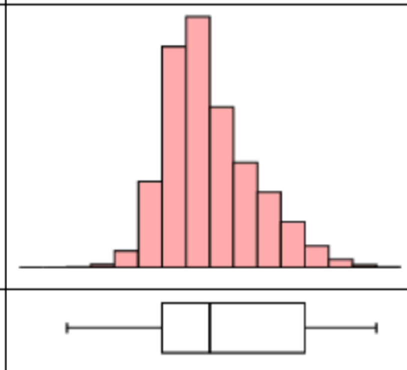
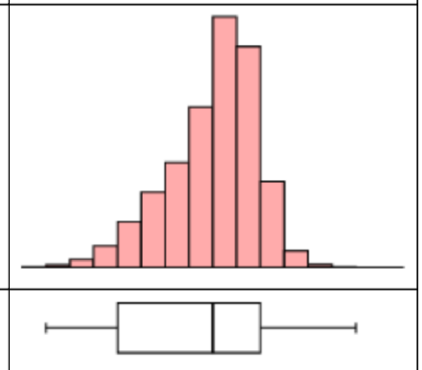
Skewness: measure of asymmetry

Left-skewed (negatively skewed): mean < median, have a few extreme small values

Right-skewed (positively skewed): mean > median, have a few extreme large values

Symmetric → mean = median (converse not true)

Symmetric + unimodal → mean = median = mode (converse not true)

Symmetric	Skewed right (positive)	Skewed left (negative)
		
Q_1 and Q_3 should be approximately equally spaced from the median (Q_2).	Q_3 is farther from the median(Q_2) than Q_1	Q_1 is farther from the median(Q_2) than Q_3

Range: maximum – minimum ($X_{(n)} - X_{(1)}$)

Interquartile range: $V_{0.75} - V_{0.25}$

Sample variance: $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ or $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i^2 - n\bar{X}^2)$

Sample standard deviation: $SD = \sqrt{S^2}$

Graphical methods

Bar graph: use for categorical data, show the number of observations in each category

Histogram: use for quantitative data, showing the number of observations in each range

Stem-and-leaf plot: ordered the data into a tree-like structure

Boxplot: show 5 numbers (min, Q1, median, Q3, max), help locate outliers (As a rule of thumb, some people define outliers as values > Q3 + 1.5*IQR or < Q1 – 1.5*IQR)

II) Probability

Notations

Sample space: the set of all possible outcomes, often denoted as Ω

Outcome: a possible type of occurrence

Event: any set of outcomes of interest, can be denoted as $E \subset \Omega$

Probability (of an event): denoted by $P(E)$, always lies between 0 and 1 (both inclusive)

$$P(E) = \frac{\#\ of\ outcomes\ in\ E}{\#\ of\ outcomes\ in\ \Omega}$$

Union: either A or B occurs, or they both occurs, denoted by $A \cup B$ (logically equivalent to OR)

Intersection: both A and B occur, denoted by $A \cap B$ (logically equivalent to AND)

Complement: A does not occur, denoted by A^C (logically equivalent to NOT)

Commutativity: $A \cup B = B \cup A,\ \ A \cap B = B \cap A$

Associativity: $(A \cup B) \cup C = A \cup (B \cup C),\ \ (A \cap B) \cap C = A \cap (B \cap C)$

Distributive laws: $(A \cap B) \cup C = (A \cup C) \cap (B \cup C),\ \ (A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

DeMorgan's laws: $(A \cup B)^C = A^C \cap B^C, (A \cap B)^C = A^C \cup B^C$

Probability theory

Mutually exclusive: A and B are mutually exclusive if $P(A \cap B) = 0$ (cannot co-occur)

Independence: $P(A \cap B) = P(A)P(B)$ iff A and B are independent

Their complements (A and B^C; A^C and B; A^C and B^C) will be pairwise independent as well

Mutual independence: $P(A \cap B \cap C) = P(A)P(B)P(C)$ iff A, B and C are mutually independent

Mutual independence implies pairwise but not vice versa

Addition law: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Multiplication law: if $A_1, \dots, A_k$ are mutually independent, then $P(A_! \cap \dots \cap A_k) = P(A_1) \times \dots \times P(A_k)$

Conditional probability

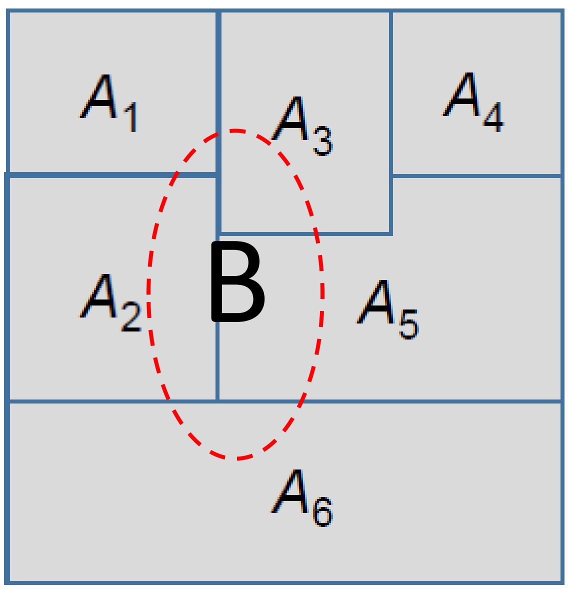
Conditional probability: $P(B|A) = \frac{P(A \cap B)}{P(A)}$, if $P(B|A) = P(B)$, then A and B are independent

Relative risk: $RR(B|A) = \frac{P(B|A)}{P(B|A^C)}$

Total probability rule: $P(B) = P(B|A)P(A) + P(B|A^C)P(A^C)$

Exhaustive: if $A_1, \ldots, A_k$ are exhaustive, then $A_1 \cup \ldots \cup A_k = \Omega$ (at least one of them must occur)

Generalized total probability rule: let $A_1, \ldots, A_k$ be mutually exclusive and exhaustive events. For any event B, we have $P(B) = \sum_{i=1}^{k} P(B|A_i)P(A_i)$



Bayes' theorem: conditional probability + generalized total probability rule. let $A_1, \ldots, A_k$ be mutually exclusive and exhaustive events. For any event B,

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{k} P(B|A_j)P(A_j)}$$

III) Discrete Probability Distributions

Random variables: numeric quantities that take different values with specified probabilities

Discrete random variable: a R.V. that takes value from a discrete set of numbers

Continuous random variable: a R.V. that takes value over an interval of numbers

Discrete random variables

Probability mass function: a pmf assigns a probability to each possible value x of the discrete random variable X, denoted by $f(x) = P(X = x)$

$\sum_{i=1}^{n} f(x_i) = 1$ (total probability rule)

Cumulative distribution function: a cdf gives the probability that X is less than or equal to the value x, denoted by $F(x) = P(X \leq x)$

Expected value: $\mu = E(X) = \sum_{i=1}^{n} x_i P(X = x_i)$ (the idea is "probability weighted average")

Variance: $\sigma^2 = Var(X) = \sum_{i=1}^{n} (x_i - \mu)^2 P(X = x_i)$ (the idea is "probability weighted distance from mean")

Alternatively $Var(X) = E(X^2) - [E(X)]^2$

Translation/rescale: $E(aX + b) = aE(X) + b, Var(aX + b) = a^2 Var(X)$

Linearity of expectation: $E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i)$

Variance of sum under independence: $Var(X + Y) = Var(X) + Var(Y)$ if X, Y are independent

Binomial distribution

Factorial: $n! = n \times (n-1) \times \ldots \times 1$, note that $0! = 1$

Permutation (order is important): $P_k^n = \frac{n!}{(n-k)!}$

Combination (order is not important): $C_k^n = \frac{n!}{k!(n-k)!}$, also denoted as $\binom{n}{k}$

Binomial distribution: probability distribution on the number of successes X in n independent experiments, each experiment has a probability of success p, then $X \sim B(n, p)$

Pmf: $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0, 1, 2, \ldots, n$

Mean: $E(X) = np$

Variance: $Var(X) = np(1-p)$

Skewness: right-skewed if p<0.5, symmetric if p=0.5, left-skewed if p>0.5

Poisson distribution

Poisson distribution: probability distribution on the number of occurrence X (usually of a rare event) over a period of time or space with rate μ, then $X \sim Po(\mu)$

Pmf: $P(X = x) = \frac{e^{-\mu}\mu^x}{x!}$ for $x = 0, 1, 2, \ldots$

Mean: $E(X) = \mu$

Variance: $Var(X) = \mu$

Skewness: right-skewed

Poisson limit theorem (poisson approximation to binomial): if $X \sim B(n, p)$ where $n \geq 20$, $p < 0.1$ and $np < 5$, then $X \approx Y \sim Po(\mu)$ where $\mu = np$

Hypergeometric distribution (not required)

Hypergeometric distribution: probability distribution on the number of success X in n trials without replacement, from a finite population of size $N_1 + N_2 = N \geq n$ that contains N_1 trials classified as success, then $X \sim Hypergeometric(N_1, N_2, n)$

Pmf: $P(X = x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}}$ for $x = \max(0, n - N_2), \ldots, \min(n, N_1)$

Mean: $E(X) = n\left(\frac{N_1}{N}\right)$

Variance: $Var(X) = n\left(\frac{N_1}{N}\right)\left(\frac{N_2}{N}\right)\left(\frac{N-n}{N-1}\right)$

Geometric distribution (not required)

Geometric distribution: probability distribution on the number of trials X when the first success occurs, each trial has a probability of success p, then $X \sim Geo(p)$

Pmf: $P(X = x) = (1-p)^{x-1}p$ for $x = 1, 2, \ldots$

Mean: $E(X) = \frac{1}{p}$

Variance: $Var(X) = \frac{1-p}{p^2}$

Memoryless: $P(X > k + j | X > k) = P(X > j)$. Geometric distribution is the only discrete distribution with this property

Negative binomial distribution (not required)

Negative binomial distribution: probability distribution on the number of times X when the r success occurs, each trial has a probability of success p, then $X \sim NB(r, p)$

Pmf: $P(X = x) = \binom{x-1}{r-1}(1-p)^{x-r}p^r$ for $x = r, r+1, \ldots$

Mean: $E(X) = \frac{r}{p}$

Variance: $Var(X) = \frac{r(1-p)}{p^2}$

IV) Continuous Probability Distributions

Continuous random variables

Probability density function: a pdf specifies the probability of a random variable falling within a particular range of values, denoted by $f(x)$

$P(a \leq X \leq b) = \int_a^b f(x)dx$, which is the area under the curve from a to b

$P(X = a) = \int_a^a f(x)dx = 0$ for all a

$\int_{-\infty}^{\infty} f(x)dx = 1$ (total probability rule)

Cumulative distribution function: a cdf gives the probability that X is less than or equal to the value x, denoted by $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$

$P(a \leq X \leq b) = \int_a^b f(x)dx = F(b) - F(a)$ (by the fundamental theorem of calculus)

Expected value: $\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx$ (the idea is "probability weighted average")

Variance: $\sigma^2 = Var(X) = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$ (the idea is "probability weighted distance from mean")

(Note: Calculus is NOT required in our course)

Translation/rescale: $E(aX + b) = aE(X) + b$, $Var(aX + b) = a^2 Var(X)$

Linearity of expectation: $E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i)$

Variance of sum under independence: $Var(X + Y) = Var(X) + Var(Y)$ if X, Y are independent

Uniform distribution

Uniform distribution: if X follows uniform distribution on the interval $[a, b]$, then it has the same probability density at any point in the interval and we denote it by $X \sim U(a, b)$

Pdf: $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$, otherwise 0

Cdf: $F(x) = \int_a^x \frac{1}{b-a} dt = \left[\frac{t}{b-a}\right]_a^x = \frac{x-a}{b-a}$ for $a \leq x \leq b$

Mean: $E(X) = \frac{a+b}{2}$

Variance: $Var(X) = \frac{(b-a)^2}{12}$

Normal distribution

Normal distribution: if X follows normal distribution with mean μ and variance σ^2, then $X \sim N(\mu, \sigma^2)$, often used to represent continuous random variable with unknown distributions

Pdf: $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ for $-\infty < x < \infty$

Shape: bell-shape, symmetric about the mean, unimodal

Standard normal distribution: $Z \sim N(0,1)$

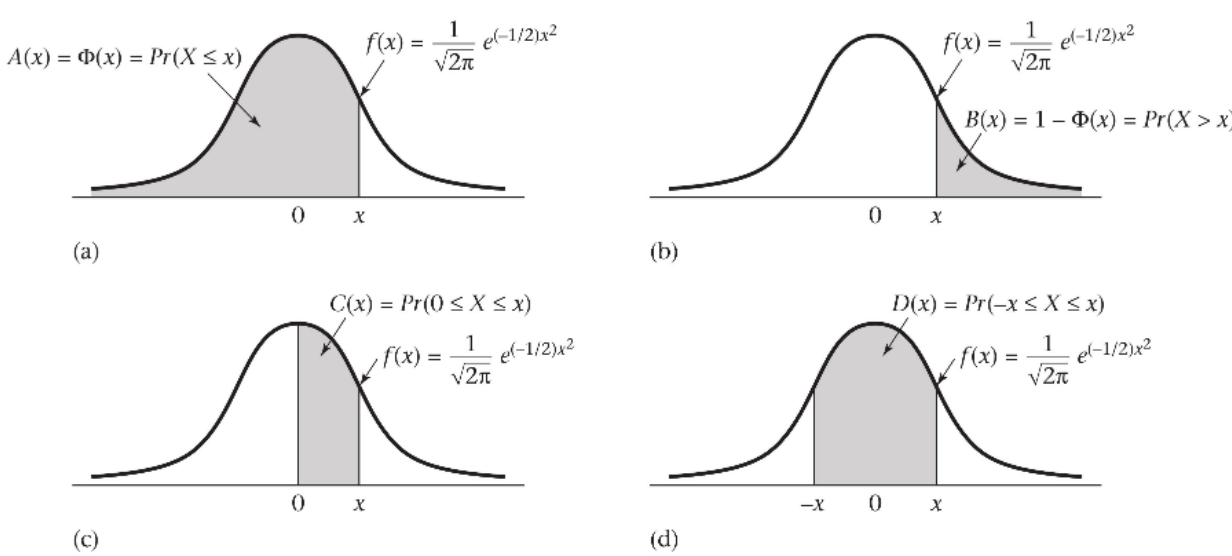
Cdf of standard normal: denoted as $\Phi(z) = P(Z \leq z)$

$P(a \leq Z \leq b) = P(Z \leq b) - P(Z \leq a) = \Phi(b) - \Phi(a)$

$\Phi(-z) = 1 - \Phi(z)$ by symmetric property

Percentile of standard normal: $\Phi(1.645) = 0.95, \Phi(1.96) = 0.975$

Standardization: if $X \sim N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0,1)$



$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$

De Moivre–Laplace theorem (normal approximation to binomial): if $X \sim B(n,p)$, $P(a < X < b) \approx P(a - 0.5 \leq Y \leq b + 0.5)$ where $Y \sim N(np, np(1-p))$. The 0.5s are continuity correction

Condition for good approximation: $np(1-p) \geq 5$

Normal approximation to poisson: if $X \sim Po(\lambda), P(X \leq a) \approx P(Y \leq a + 0.5)$ where $Y \sim N(\lambda, \lambda)$

Condition for good approximation: $\lambda \geq 10$

Some remarks (not required)

Statistical parameter: a numerical characteristic of a statistical population or a statistical model. We are given these numbers (e.g. p, λ, μ) in previous chapters but in reality we do not know these numbers. These lead to the next part of our course: Statistical Inference

Why approximation: one major reason is that calculating binomial probability involves combination and large factorials are hard/costly to compute in previous centuries

Variance of sum: $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X, Y)$

Tower rule of expectation: $E(X) = E[E(X|Y)]$

Law of total variance (EVE): $Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$

Sum of poisson: if $X \sim Po(\lambda_1), Y \sim Po(\lambda_2)$ independently, then $X + Y \sim Po(\lambda_1 + \lambda_2)$

Sum of normal: if $X \sim N(\mu_1, \sigma_1^2), Y \sim N(\mu_2, \sigma_2^2)$ independently, then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Square of standard normal: if $X \sim N(\mu, \sigma^2)$, the $Z^2 = \left[\frac{X-\mu}{\sigma}\right]^2 \sim \chi_1^2$

Sum of chi square: if $X \sim \chi_n^2, Y \sim \chi_m^2$, then $X + Y \sim \chi_{n+m}^2$

V) Point Estimation

Statistical inference: process of drawing conclusions from data that are subject to random variations

Estimation: estimate the values of specific population parameters based on the observed data

Hypothesis testing: test on whether the value of a population parameter is equal to some specific value based on the observed data

Sampling

Sample: the data obtained after the experiments are performed, usually denoted by $x_1, \dots, x_n$

Random sample: the data before the experiments are performed, usually denoted by $X_1, \dots, X_n$

Non-probability sample: some elements of the population have no chance of being selected

Probability sample: all elements in the population has known nonzero chance to be selected

Simple random sample: all elements in the population has the same probability to be selected

Systematic sample: elements are selected at regular intervals through certain order

Stratified sample: all elements are classified into different stratums and each stratum is sampled as an independent sub-population

Cluster sample: all elements are divided into different clusters and a simple random sample of clusters is selected

Coverage error: exists if some groups are excluded from the frame and have no chance of being selected

Non-response error: people who do not respond may be different from those who do respond

Measurement error: due to weaknesses in question design, respondent error, and interviewer's impact on the respondent

Sampling error: Chance (luck of the draw) variation from sample to sample

Point estimator

Point estimator: a rule for calculating a single value to "best guess" an unknown population parameter of interest based on the observed data

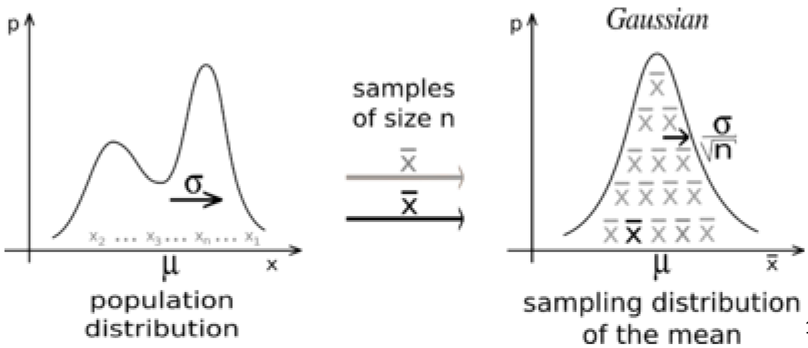
(Note: estimator $\hat{\theta}(X)$ is random, estimate $\hat{\theta}(x)$ is fixed, estimand θ is the unknown parameter)

Unbiasedness: $E(\hat{\theta}) = \theta$

Minimum variance: $Var(\hat{\theta}) \leq Var(\tilde{\theta})\ \ \forall\, \tilde{\theta} \in \Theta$

Independent and identically distributed (i.i.d.): an assumption where the random variables $X_1, \ldots, X_n$ are sampled such that they are independent and follows the same distribution

Central limit theorem (CLT, Lindeberg–Lévy): Let $X_1, \ldots, X_n$ be i.i.d. random variables with mean μ and finite variance σ^2, then as n tends to infinity (>30 in practice), $\bar{X} \approx N\left(\mu, \frac{\sigma^2}{n}\right)$



[1]

Mean

Estimand: $\theta = \mu = E(X)$

Sample mean (estimator): $\hat{\theta} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

Expectation: $E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{n\mu}{n} = \mu$ (unbiased)

Variance: $Var(\bar{X}) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$ (by i.i.d.)

Distribution: $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$. If $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$, then this follows from the fact that sum of independent normal is normal (remarks in section IV).

If $X_1, \ldots, X_n$ follows some other distribution, then this follows from the CLT when n is large (usually >30). Otherwise ($n \leq 30$) we have $\sqrt{n}\left(\frac{\bar{X}-\mu}{S}\right) \sim t_{n-1}$, where t_{n-1} is a Student's t-distribution with degree of freedom n-1.

[1] By Mathieu ROUAUD - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=60066898

Variance

Estimand: $\theta = \sigma^2 = Var(X)$

Sample variance (estimator): $\hat{\theta} = S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$, $S'^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$ if μ is known

Expectation: $E(S^2) = \sigma^2$ (unbiased)

Variance: $Var(S^2) = \frac{2\sigma^4}{n-1}$ (not required)

Distribution: $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \Rightarrow S^2 \sim \frac{\sigma^2}{n-1}\chi^2_{n-1}$ (right-skewed)

Binomial proportion

Estimand: $\theta = p = E(Y)$ where $Y_1, \dots, Y_n \sim B(1, p)$ (similar to mean case)

Estimator: $\hat{p} = \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$

Expectation: $E(\hat{p}) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i) = \frac{np}{n} = p$ (unbiased)

Variance: $Var(\hat{p}) = \frac{1}{n^2}\sum_{i=1}^{n} Var(Y_i) = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$ (by i.i.d.)

Distribution: $\hat{p} \sim B(n, p)$ because the sampling distribution is binomial. For $n > 30$ or $n\hat{p}\hat{q} > 5$, normal approximation gives $\hat{p} \sim N\left(p, \frac{p(1-p)}{n}\right)$

Poisson rate

Estimand: $\theta = \lambda$ where $X \sim Po(\lambda T)$ with T as the total number of units

Estimator: $\hat{\lambda} = \frac{X}{T}$

Expectation: $E(\hat{\lambda}) = \frac{1}{T}E(X) = \frac{\lambda T}{T} = \lambda$ (unbiased)

Variance: $Var(\hat{\lambda}) = \frac{1}{T^2}Var(X) = \frac{\lambda T}{T^2} = \frac{\lambda}{T}$ (by i.i.d.)

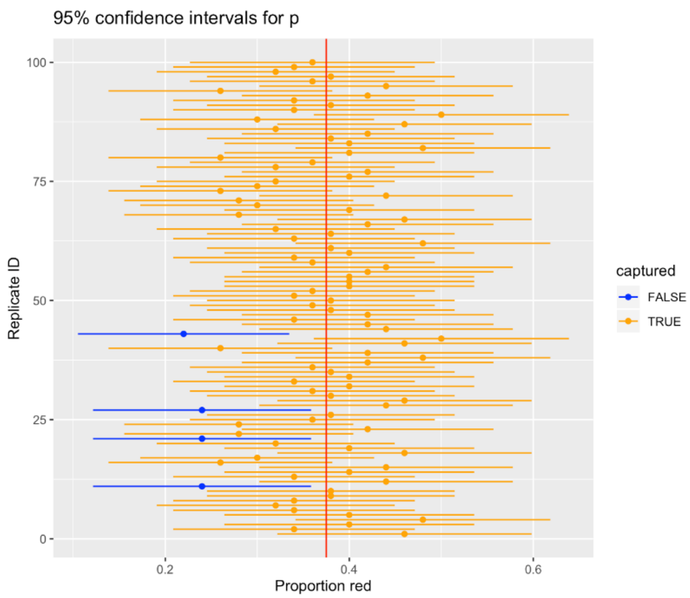
Distribution: $\hat{\lambda} \sim Po(\lambda T)$ because the sampling distribution is Poisson. For $n > 30$ or $\hat{\lambda}T > 10$, normal approximation gives $\hat{\lambda} \sim N\left(\lambda, \frac{\lambda}{T}\right)$

VI) Interval Estimation

Confidence interval

Confidence interval: an interval associated with a confidence level $1-\alpha$ that may contain the true value of an unknown population parameter

Meaning of confidence level: in the long run, $100(1-\alpha)\%$ of all the confidence intervals that can be constructed will contain the unknown true parameter (NOT the probability that an interval will contain the parameter)



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Elements of confidence interval: $\{\hat{\theta}, c_{\alpha}, se(\hat{\theta})\}$, where $\hat{\theta}$ is the point estimate, c_a is the critical value from an asymptotic distribution under the confidence level $1-\alpha$, $se(\hat{\theta})$ is the standard error of the point estimate

Mean

Confidence interval (σ is known): $\bar{x} \pm z_{\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$

Confidence interval (σ is unknown, $n > 30$): $\bar{x} \pm z_{\frac{\alpha}{2}} \times \frac{s}{\sqrt{n}}$

Confidence interval (σ is unknown, $n \leq 30$): $\bar{x} \pm t_{n-1,\frac{\alpha}{2}} \times \frac{s}{\sqrt{n}}$ (differs in degree of freedom)

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[2] By Chester Ismay and Albert Y. Kim from Ch9 Confidence Intervals of Statistical Inference via Data Science

Margin of error: $E = c_\alpha \times se(\hat{\theta})$ (width is $2E$ which helps determine sample size)

Critical values: standard normal and t-distribution are symmetric around 0 $\Rightarrow c_{1-\frac{\alpha}{2}} = c_{\frac{\alpha}{2}}$

Common normal critical value: $z_{0.95} = 1.645, z_{0.975} = 1.96, z_{0.995} = 2.575$

One-sided confidence interval: $\mu > \bar{x} - z_{1-\alpha} \times \frac{\sigma}{\sqrt{n}}$ or $\mu < \bar{x} + z_{1-\alpha} \times \frac{\sigma}{\sqrt{n}}$

(Note: this is essentially adjusting the critical value, which arises naturally when we are not interested in the other bound, e.g. weight > 0 so negative lower bound is not interested)

Variance

Confidence interval (μ is unknown): $\left(\frac{(n-1)s^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \frac{(n-1)s^2}{\chi^2_{n-1,\frac{\alpha}{2}}}\right)$

Confidence interval (μ is known): $\left(\frac{ns'^2}{\chi^2_{n,1-\frac{\alpha}{2}}}, \frac{ns'^2}{\chi^2_{n,\frac{\alpha}{2}}}\right) = \left(\frac{\sum_{i=1}^n (X_i-\mu)^2}{\chi^2_{n,1-\frac{\alpha}{2}}}, \frac{\sum_{i=1}^n (X_i-\mu)^2}{\chi^2_{n,\frac{\alpha}{2}}}\right)$ (differs in d.f.)

Critical values: chi-squared distribution is not symmetric, so cannot simplify

Binomial proportion

Confidence interval ($n > 30$ or $n\hat{p}\hat{q} > 5$): $\hat{p} \pm z_{\frac{\alpha}{2}} \times se(\hat{p}) \approx \hat{p} \pm z_{\frac{\alpha}{2}} \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

(Note: the standard error here is an approximated version from the lecture notes)

Confidence interval (exact method): solve for p_L, p_U from $\begin{cases} P(X \geq n\hat{p} | p = p_L) = \frac{\alpha}{2} \\ P(X \leq n\hat{p} | p = p_U) = \frac{\alpha}{2} \end{cases}$ where $X \sim B(n, p)$

Poisson rate

Confidence interval (exact method): solve for λ_L, λ_U from $\begin{cases} P\left(X \geq \hat{\lambda}T \middle| \lambda = \lambda_L\right) = \frac{\alpha}{2} \\ P\left(X \leq \hat{\lambda}T \middle| \lambda = \lambda_U\right) = \frac{\alpha}{2} \end{cases}$ where $X \sim Po(\lambda T)$

Confidence interval (bootstrap method): generate N sample of size m with replacement from X. Calculate the point estimate from each bootstrap sample. Sort the means and the bootstrap confidence interval is given by the corresponding percentiles.

(Note: bootstrap is a very powerful method which can be applied to many statistical problems that do not require close form)

VII) Hypothesis Testing

Terminologies

Statistical hypothesis: a claim (assumption) about a population parameter

Null hypothesis: H_0, the hypothesis to be tested (default position)

Alternative hypothesis: H_1, a hypothesis challenge (against) H_0 (what we want to conclude)

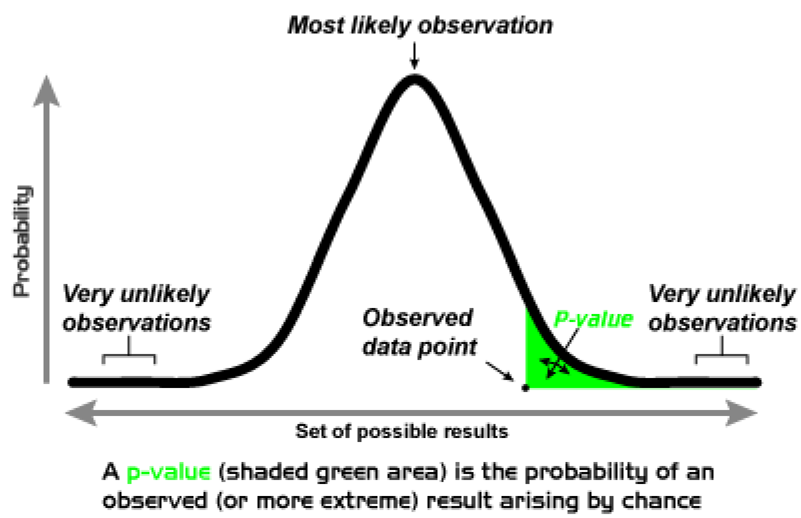
Hypothesis testing: a procedure to make decision on hypothesis based on some data samples. The idea is to assume H_0 is true first. If the population under H_0 is unlikely to generate the data sample, then we can make a decision to reject H_0 (and thus accept H_1).

Test statistics: a quantity (statistics) derived from the sample to help perform hypothesis test

Level of significance: α, defines the unlikely value of the sample if H_0 is true

Critical value: cutoff values from the distribution of test statistic under H_0 given α

p-value: probability of obtaining a test statistics at least as extreme as the observed sample value given H_0 is true



[3]

"Accept the null hypothesis": if we fail to reject H_0, we cannot accept it because doing so violates the idea of prove by contradiction. It is possible that H_0 is not true but we have not collected enough data to reject it

Type I error: α, reject H_0 when H_0 is true (false positive).

[3] By Repapetilto - Adobe IllustratorPreviously published: Unpublished, CC BY-SA 3.0, https://en.wikipedia.org/w/index.php?curid=35569621

(Note: traditional statistical procedure controls type I error by the level of significance, so that's why both of them are α)

Type II error: β, do not reject H_0 when H_0 is false (false negative)

	H_0 is true	H_0 is false
Do not reject H_0	Correct inference (true negative, probability = 1-α)	Type II error (false negative, probability = β)
Reject H_0	Type I error (false positive, probability = α)	Correct inference (true positive, probability = 1-β)

Duality of confidence interval with hypothesis test: H_0 is rejected at significance level α if and only if the corresponding confidence interval does not contain the value claimed by H_0 with confidence level $1-\alpha$ (true for common tests)

One-sample z-test

Assumption: known σ, from normal distribution or of large size ($n \geq 30$)

Hypothesis: (1) $\begin{cases} H_0: \mu = \mu_0 \\ H_1: \mu \neq \mu_0 \end{cases}$ or (2) $\begin{cases} H_0: \mu = \mu_0 \\ H_1: \mu > \mu_0 \end{cases}$ or (3) $\begin{cases} H_0: \mu = \mu_0 \\ H_1: \mu < \mu_0 \end{cases}$

Test statistics: $z_0 = \frac{\bar{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}}$, $Z_0 \sim N(0,1)$ under null

Note: the capital Z_0 is not typo but indicates that it is random

Decision rule: reject if (1) $|z_0| > z_{1-\frac{\alpha}{2}}$; (2) $z_0 > z_{1-\alpha}$; (3) $z_0 < z_\alpha$

p-value: reject if $p_0 < \alpha$ where (1) $p_0 = P(Z_0 > |z_0|)$; (2) $p_0 = P(Z_0 > z_0)$; (3) $p_0 = P(Z_0 < z_0)$

One-sample t-test

Assumption: unknown σ

Hypothesis: (1) $\begin{cases} H_0: \mu = \mu_0 \\ H_1: \mu \neq \mu_0 \end{cases}$ or (2) $\begin{cases} H_0: \mu = \mu_0 \\ H_1: \mu > \mu_0 \end{cases}$ or (3) $\begin{cases} H_0: \mu = \mu_0 \\ H_1: \mu < \mu_0 \end{cases}$

Test statistics: $t_0 = \frac{\bar{x}-\mu_0}{\frac{s}{\sqrt{n}}}$, $T_0 \sim t_{n-1}$ under null

Decision rule: reject if (1) $|t_0| > t_{n-1,1-\frac{\alpha}{2}}$; (2) $t_0 > t_{n-1,1-\alpha}$; (3) $t_0 < t_{n-1,\alpha}$

One-sample chi-squared test

Assumption: unknown σ, from normal distribution

Hypothesis: (1) $\begin{cases} H_0: \sigma^2 = \sigma_0^2 \\ H_1: \sigma^2 \neq \sigma_0^2 \end{cases}$ or (2) $\begin{cases} H_0: \sigma^2 = \sigma_0^2 \\ H_1: \sigma^2 > \sigma_0^2 \end{cases}$ or (3) $\begin{cases} H_0: \sigma^2 = \sigma_0^2 \\ H_1: \sigma^2 < \sigma_0^2 \end{cases}$

Test statistics: $\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2}$, $\mathrm{X}_0^2 \sim \chi_{n-1}^2$ under null

Decision rule: reject if (1) $\chi_0^2 > \chi_{n-1,1-\frac{\alpha}{2}}^2$ or $\chi_0^2 < \chi_{n-1,\frac{\alpha}{2}}^2$; (2) $\chi_0^2 > \chi_{n-1,1-\alpha}^2$; (3) $\chi_0^2 < \chi_{n-1,\alpha}^2$

One-sample binomial proportion test

Assumption: binomial sample with $n > 30$ or $np_0q_0 > 5$

Hypothesis: (1) $\begin{cases} H_0: p = p_0 \\ H_1: p \neq p_0 \end{cases}$ or (2) $\begin{cases} H_0: p = p_0 \\ H_1: p > p_0 \end{cases}$ or (3) $\begin{cases} H_0: p = p_0 \\ H_1: p < p_0 \end{cases}$

Test statistics: $z_0 = \frac{\bar{y} - p_0}{\frac{\sqrt{p_0(1-p_0)}}{\sqrt{n}}}$, $Z_0 \sim N(0,1)$ under null

Decision rule: reject if (1) $|z_0| > z_{1-\frac{\alpha}{2}}$; (2) $z_0 > z_{1-\alpha}$; (3) $z_0 < z_\alpha$

Some remarks (not required)

Power: $P(reject\ H_0 | H_1\ is\ true)$. As higher power implies a lower type II error, traditional procedures usually fix the type I error and search for tests with high power

Bayesian inference: most procedures in this course are frequentist procedures. Taking interval estimation as an example, if we want our interval to have probability $1 - \alpha$ covering the unknown parameter, we should seek credible interval from Bayesian inference instead (confidence interval does not guarantee that). Consider taking more courses from our department if you are interested :)

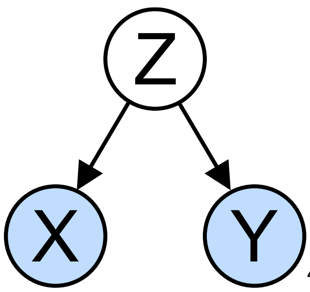
VIII) Extension (not required)

Terminologies

Longitudinal study: repeated observations of the same variables over a period of time

Cross-sectional study: observations from a population at a specific point in time

Confounder: a variable that influences both the dependent variable and independent variable



[4]

Difference of mean, two dependent samples

Assumption: both from normal, of large size ($n \geq 30$) or difference is approximately normal

Hypothesis: (1) $\begin{cases} H_0: \Delta = 0 \\ H_1: \Delta \neq 0 \end{cases}$ or (2) $\begin{cases} H_0: \Delta = 0 \\ H_1: \Delta > 0 \end{cases}$ or (3) $\begin{cases} H_0: \Delta = 0 \\ H_1: \Delta < 0 \end{cases}$ where $\Delta = \mu_X - \mu_Y$

Z-test (known σ): $z_0 = \frac{\bar{d}-\Delta}{\frac{\sigma_3}{\sqrt{n}}}$, $\bar{D} \sim N\left(\Delta, \frac{\sigma_3^2}{n} = \frac{\sigma_X^2+\sigma_Y^2}{n}\right)$ under null

Decision rule: reject if (1) $|z_0| > z_{1-\frac{\alpha}{2}}$; (2) $z_0 > z_{1-\alpha}$; (3) $z_0 < z_\alpha$

T-test (unknown σ): $t_0 = \frac{\bar{d}-\Delta}{\frac{s}{\sqrt{n}}}$, $T_0 \sim t_{n-1}$ under null, $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(d_i - \bar{d}\right)^2$

Decision rule: reject if (1) $|t_0| > t_{n-1,1-\frac{\alpha}{2}}$; (2) $t_0 > t_{n-1,1-\alpha}$; (3) $t_0 < t_{n-1,\alpha}$

Confidence interval (σ is known): $\bar{d} \pm z_{\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$

Confidence interval (σ is unknown, $n > 30$): $\bar{d} \pm z_{\frac{\alpha}{2}} \times \frac{s}{\sqrt{n}}$

Confidence interval (σ is unknown, $n \leq 30$): $\bar{d} \pm t_{n-1,\frac{\alpha}{2}} \times \frac{s}{\sqrt{n}}$ (differs in degree of freedom)

Very similar to one-sample case due to duality of CI and testing

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[4] By طاها - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=31197358

Difference of mean, two independent samples

Assumption: both from normal or of large size ($n, m \geq 30$ though can be different)

Hypothesis: (1) $\begin{cases} H_0: \Delta = 0 \\ H_1: \Delta \neq 0 \end{cases}$ or (2) $\begin{cases} H_0: \Delta = 0 \\ H_1: \Delta > 0 \end{cases}$ or (3) $\begin{cases} H_0: \Delta = 0 \\ H_1: \Delta < 0 \end{cases}$ where $\Delta = \mu_X - \mu_Y$

Z-test (known σ_X, σ_Y): $z_0 = \frac{\bar{x}-\bar{y}-\Delta}{\sqrt{\frac{\sigma_X^2}{n}+\frac{\sigma_Y^2}{m}}}$, $Z_0 \sim N(0,1)$ under null

Decision rule: reject if (1) $|z_0| > z_{1-\frac{\alpha}{2}}$; (2) $z_0 > z_{1-\alpha}$; (3) $z_0 < z_\alpha$

Approximate z-test (unknown σ_X, σ_Y but $n, m \geq 30$): $z_0 = \frac{\bar{x}-\bar{y}-\Delta}{\sqrt{\frac{s_X^2}{n}+\frac{s_Y^2}{m}}}$, , $Z_0 \sim N(0,1)$ under null

Decision rule: reject if (1) $|z_0| > z_{1-\frac{\alpha}{2}}$; (2) $z_0 > z_{1-\alpha}$; (3) $z_0 < z_\alpha$

T-test (unknown $\sigma_X = \sigma_Y$; $n, m < 30$; both from normal): $t_0 = \frac{\bar{x}-\bar{y}-\Delta}{s_p\sqrt{\frac{1}{n}+\frac{1}{m}}}$, $T_0 \sim t_{n+m-2}$ under null

Pooled variance estimate: $s_p^2 = \frac{(n-1)s_X^2+(m-1)s_Y^2}{n+m-2}$, $E\left(S_p^2\right) = \sigma_X^2 = \sigma_Y^2$ by assumption

Decision rule: reject if (1) $|t_0| > t_{n+m-2,1-\frac{\alpha}{2}}$; (2) $t_0 > t_{n+m-2,1-\alpha}$; (3) $t_0 < t_{n+m-2,\alpha}$

T-test (unknown σ_X, σ_Y; $n, m < 30$; both from normal)): $t_0 = \frac{\bar{x}-\bar{y}-\Delta}{\sqrt{\frac{s_X^2}{n}+\frac{s_Y^2}{m}}}$, $T_0 \sim t_{d'}$ under null

Satterthwaite's method: $d' = \frac{\frac{s_X^2}{n}+\frac{s_Y^2}{m}}{\frac{\left(\frac{s_X^2}{n}\right)^2}{n-1}+\frac{\left(\frac{s_Y^2}{m}\right)^2}{m-1}}$, which should fall between $n-1, m-1$

Decision rule: reject if (1) $|t_0| > t_{d',1-\frac{\alpha}{2}}$; (2) $t_0 > t_{d',1-\alpha}$; (3) $t_0 < t_{d',\alpha}$

Behrens-Fisher problem: when $X \sim N(\mu_X, \sigma_X^2), Y \sim N(\mu_Y, \sigma_Y^2)$ but $\sigma_X \neq \sigma_Y$ are unknown, how to test $\mu_X = \mu_Y$?

Obviously Satterthwaite's method is one possible solution but it may not be best

Difference of proportion, two independent samples

Assumption: binomial sample with $n, m > 30$ or $np_Xq_X, mp_Yq_Y > 5$

Hypothesis: (1) $\begin{cases} H_0: \Delta = 0 \\ H_1: \Delta \neq 0 \end{cases}$ or (2) $\begin{cases} H_0: \Delta = 0 \\ H_1: \Delta > 0 \end{cases}$ or (3) $\begin{cases} H_0: \Delta = 0 \\ H_1: \Delta < 0 \end{cases}$ where $\Delta = p_X - p_Y$

Test statistics: $z_0 = \frac{\frac{x}{n} - \frac{y}{m}}{\sqrt{\bar{p}(1-\bar{p})\left(\frac{1}{n}+\frac{1}{m}\right)}}$, $Z_0 \sim N(0,1)$ under null, $\bar{p} = \frac{x+y}{n+m}$

Decision rule: reject if (1) $|z_0| > z_{1-\frac{\alpha}{2}}$; (2) $z_0 > z_{1-\alpha}$; (3) $z_0 < z_\alpha$

Ratio of variance, two independent samples

Assumption: unknown σ_X, σ_Y and both from normal independently

Hypothesis: (1) $\begin{cases} H_0: \sigma_X^2 = \sigma_Y^2 \\ H_1: \sigma_X^2 \neq \sigma_Y^2 \end{cases}$ or (2) $\begin{cases} H_0: \sigma_X^2 = \sigma_Y^2 \\ H_1: \sigma_X^2 > \sigma_Y^2 \end{cases}$ or (3) $\begin{cases} H_0: \sigma_X^2 = \sigma_Y^2 \\ H_1: \sigma_X^2 < \sigma_Y^2 \end{cases}$

Test statistics: $f_0 = \frac{s_X^2}{s_Y^2}$, $F_0 \sim F_{n-1,m-1}$ under null

Decision rule: reject if (1) $f_0 > F_{n-1,m-1,1-\frac{\alpha}{2}}$ or $f_0 < F_{n-1,m-1,\frac{\alpha}{2}}$; (2) $f_0 > F_{n-1,m-1,1-\alpha}$; (3) $f_0 < F_{n-1,m-1,\alpha}$