RMSC5102 Midterm Review

Leung Man Fung, Heman

Spring, 2021

Agenda

- Review
  - Basic Knowledge
  - The Black-Scholes World
  - Monte Carlo Method
  - Random Variable Generation
  - Variance Reduction Technique
- Q&A

Basic Knowledge

Geometric Distribution

- $X \sim \text{Geo}(p)$
  - number of Bernoulli trials X needed to get 1 success
  - each trial has a probability of success p
- Pmf: $\mathbb{P}(X = x) = (1 - p)^{x-1} p$ for $x = 1, 2, \ldots$
- Mean: $\mathbb{E}(X) = \frac{1}{p}$
- Variance: $\text{Var}(X) = \frac{1-p}{p^2}$
- Related to rejection sampling

Exponential Distribution

- $X \sim \text{Exp}(\lambda)$
  - Continuous analogue of the geometric distribution
  - We adopt the rate parametrization instead of scale
- Pdf: $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$, otherwise 0
- Cdf: $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$
- Mean: $\mathbb{E}(X) = \frac{1}{\lambda}$
- Variance: $\text{Var}(X) = \frac{1}{\lambda^2}$
- Memoryless property: $\mathbb{P}(X > s + t | X > s) = \mathbb{P}(X > t)$ for $s, t \geq 0$
  - Exponential distribution is the only continuous distribution that has this property
- Useful representation: if $Y \sim \text{Exp}(1)$, then $\frac{Y}{\lambda} \sim \text{Exp}(\lambda)$

Some Properties of Expectation, Variance and Covariance

- Law of the unconscious statistician: $\mathbb{E}[g(X)] = \sum_{i=1}^{n} g(x_i)\mathbb{P}(X = x_i)$
  - Also holds for continuous random variables
- Translation/rescale:
  - $\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$
  - $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$
  - $\mathrm{Cov}(aX + b, cY + d) = ac\mathrm{Cov}(X, Y), \mathrm{Cov}(X, X) = \mathrm{Var}(X)$
- Linearity of expectation: $\mathbb{E}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathbb{E}(X_i)$
- Alternative formula for variance: $\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$
- Example: HW1 5, HW2 1

Geometric Brownian Motion

- SDE: $\mathrm{dS_t} = rS_t dt + \sigma S_t dW_t \Rightarrow \mathrm{S_t} = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$
  - Use $S_T = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z}$ in simulation to avoid simulating intermediate prices
- Algorithm:
  - Generate $Z \sim N(0,1)$
  - Set $S_T = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z}$
- Example: HW1 1, 2; HW2 5
- Question may specify stock price dynamic other than GBM
  - Use the given dynamic to simulate S_T like generating random variables

The Black-Scholes World

Risk Neutral Valuation

- $V_t = e^{-r(T-t)}\mathbb{E}[f(S_t, t)]$
- Take expectation w.r.t. real world probability?
  - E.g., with insider info you know price of a certain stock will likely go up
- Problem of the discount rate
  - If real world probability is used, discount rate has to accommodate the level of risk (think about the discount rate you use in DCF)
  - If risk neutral probability is used, discount rate = risk free rate (observable)
  - Just give you another way of looking at risk neutral approach
- The above formula always hold. Why do we need model like GBM then?
  - Because $r, t, T, f(\cdot,\cdot)$ are known/observable but S_t is not

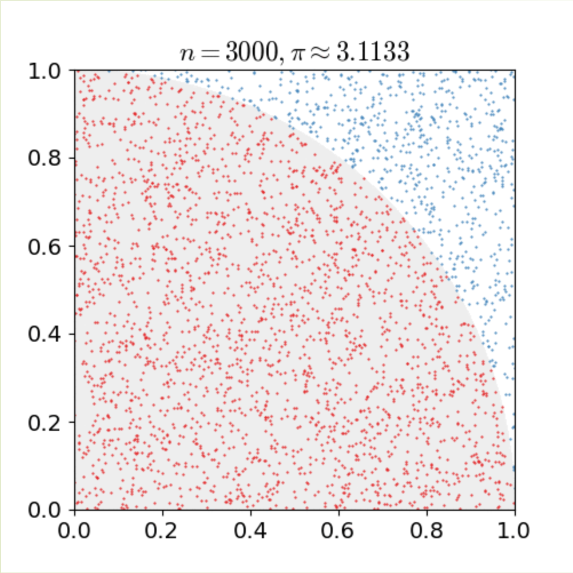
Black–Scholes–Merton Model

- Black-Scholes formula
  - $C(S_t, t) = \Phi(d_1)S_t - \Phi(d_2)Ke^{-r(T-t)}$,
  - $P(S_t, t) = Ke^{-r(T-t)} - \mathrm{S_t} + C(S_t, t) = \Phi(-d_2)Ke^{-r(T-t)} - \Phi(-d_1)S_t$
  - $d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$
  - $d_2 = d_1 - \sigma\sqrt{T-t}$
- Note that $P(S_t, t)$ is derived from put-call parity
  - Put-call parity: $C_E - P_E = S - Ke^{-r(T-t)}$
- Implied volatility
  - Value of volatility when back-solving an option pricing model (such as BS) with current market price

Monte Carlo Method

Key Idea

- Use repeated random sampling to obtain numerical estimate
  - The estimate is usually average in our course
- Example: estimate π (picture credit: nicoguaro)



Standard Monte Carlo

- HW2 5a: price a European call option
  - Recall payoff function is $\max(S_T - K, 0)$
  - Estimate $\mathbb{E}[\max(S_T - K, 0)]$ by sample average $\frac{1}{n}\sum_{i=1}^{n} \max(S_T^{(i)} - K, 0)$
- Algorithm
  - 1) Generate $Z \sim N(0,1)$
  - 2) Set $S_T = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z}$
  - **3) Compute $\boldsymbol{\pi_i = \max(S_T - K, 0)}$**
  - 4) Repeat 1 to 3 for $i = 1, \dots, n$
  - 5) Option price = $\frac{\mathrm{e^{-rT}}}{\mathrm{n}} \sum_{i=1}^{n} \pi_i$

Stopping a Simulation

- Margin of error: terminate a simulation when $\frac{s^2}{\sqrt{n}} \leq d$
  - Where s^2 is the sample variance and d is the maximum tolerable error
  - This is essentially based on the Central Limit Theorem
- Law of large numbers (WLLN): Let $X_1, \ldots, X_n$ be i.i.d. random variables with mean θ and variance σ^2, then $\bar{X}_n \approx \theta$ as $n \to \infty$
- Central limit theorem: Let $X_1, \ldots, X_n$ be i.i.d. random variables with mean θ and finite variance σ^2, then $\bar{X}_n \approx \mathrm{N}\left(\theta, \frac{\sigma^2}{n}\right)$ as $n \to \infty$

Random Variable Generation

Key ideas

- Monte Carlo methods rely on repeated random sampling
  - Need way(s) to generate different random variables
- Only $\text{Unif}(0,1)$ and $\text{N}(0,1)$ can be generated without any algorithm
  - Technically only $\text{Unif}(0,1)$ (pseudorandom number)
  - $\text{N}(0,1)$ is generated by special algorithm as well, e.g., Box–Muller transformation
  - They are omitted in this course
- Modulo operation: find the remainder of a division; denoted by mod.
  - E.g., $97 = 7 \pmod{10}$, $25 = 1 \pmod{8}$
- Use inverse transform/rejection sampling if other distribution appears

Inverse Transform

- Probability integral transform: if X is a continuous random variable with cdf F_X, then $Y = F_X(X) \sim \text{Unif}(0,1)$
- Therefore, if we know $X \sim F_X$ (i.e., the cdf), we can generate X out of $U \sim \text{Unif}(0,1)$
  - Need to derive cdf if only pdf is given
- Algorithm (discrete)
  - Generate $U \sim \text{Unif}(0,1)$
  - $X = x_j$ if $\sum_{i=0}^{j-1} p_i \leq U < \sum_{i=0}^{j} p_i$
- Algorithm (continuous)
  - Generate $U \sim \text{Unif}(0,1)$
  - $X = F_X^{-1}(U)$ assuming the inverse exists
- Example: HW2 1c, 4a

Rejection Sampling

- If we can simulate $Y \sim G_Y$ easily, we can use the proportional distribution as a basis to simulate X with pdf $f(x)$
- Algorithm
  - 1) Find $c = \max_y \frac{f(y)}{g(y)}$
  - 2) Generate Y_i from a density g: $U_1 \sim \text{Unif}(0,1) \Rightarrow Y_i = G^{-1}(U_1)$
  - 3) Generate $U_2 \sim \text{Unif}(0,1)$
  - 4) If $\text{U}_2 \leq \frac{1}{c} \cdot \frac{f(Y_i)}{g(Y_i)}$, set $X_i = Y_i$, otherwise return to 2
- Example: HW2 3, 4b
- Inverse transform is rejection sampling with $c = 1$
  - Because inverse transform simulate from F directly (always accept)

Variance Reduction Technique

Antithetic Variables

- If we are able to generate negatively correlated underlying random variables, the estimator can have lower variance as compared with independent samples
  - This requires the target function $h(x)$ to be monotone
  - Show $h'(x) \geq 0$ or $h'(x) \leq 0$ within the target range for monotonicity
  - As $h(x)$ is monotone, $\mathrm{Cov}[h(U), h(1-U)] \leq 0$ where $U \sim \mathrm{Unif}(0,1)$
  - As half of your variables are antithetic, you only need to generate $\frac{n}{2}$ numbers for n samples

Antithetic Variables

- Algorithm:
  - 1) Generate $U \sim \text{Unif}(0,1)$
  - 2) Set $X_i = F^{-1}(U), Y_i = F^{-1}(1-U)$ (note: want X, Y same distribution but negative correlation)
  - 3) Repeat 1 and 2 for n times
  - 4) $\hat{\theta} = \frac{1}{2n}\sum_{i=1}^{n}[h(X_i) + h(Y_i)]$
- Note:
  - $F^{-1}(U)$ is monotone in general as cdf is monotone
  - Hence $h[F^{-1}(U)]$ is monotone if $h(\cdot)$ is monotone


IF YOURE GOING THROUGH HELL
KEEP GOING
-Winston Churchill
Karyera.c

Q&A